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LETTER TO THE EDITOR

The quantum double and the universal R matrix for non-standard deformation of $A_{(n-1)}$

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Abstract. The quantum double and the universal R matrix for a new quantum group associated with a non-standard solution of the Yang–Baxter equation is constructed. This quantum group can be regarded as a generalization of $U_q(sl(n)$ and our construction reduces to that of $U_q(sl(n))$ for a special choice of parameters.

In a recent letter [1] we studied the quantum group associated with a generalization of the R matrix of $sl(n)$, which reads:

$$R=\sum_{i\neq j}^{n} e_{ii}\otimes e_{jj}+\sum_{i=1}^{n} q_i e_{ii}\otimes e_{ii}+(q-q^{-1})\sum_{i<j}^{n} e_{ji}\otimes e_{ij} \tag{1}$$

where each q_i can independently be equal to q or $-q^{-1}$.

The possibility of setting each individual q_i equal to $-q^{-1}$, leads to the appearance of nilpotent elements in this new quantum group which was denoted by $X_q(A_{n-1})$. Nilpotency of roots was the main puzzling feature of this quantum group which was the sign of some hidden superstructure. We also gave the multiparametric form of the R matrix (1) and showed that the additional parameters, by a suitable ansatz for the generators of the quantum group leads only to the twisting [2] of the Hopf structure. However the question of the quantum double and [3] and the universal R matrix was left unanswered in our previous paper. In this letter we report on the full quantum double construction and the universal R matrix for $X_q(A_{n-1})$. Due to the technical difficulties for constructing the quantum double, only a few concrete examples are available to this date [4–7]. This letter is aimed to add a new concrete example to the above list.

We now consider the structure of $X_q(A_{n-1})$ in the Chevalley basis. In [1], $X_q(A_{n-1})$ was constructed by applying the FRT formalism to the R matrix (1) with the following ansatz of the matrices L^+ and L^- in the equations $RL_2^{\pm}L_1^{\pm}=L_1^{\mp}L_2^{\pm}R$, etc).

$$L^+=\sum_{i=1}^{n} k_i e_{ii}+\sum_{i=1}^{n} w\left(\frac{q_i}{q_{i+1}}\right)^{1/4}(k_ik_{i+1})^{1/2}X_i^+e_{i,i+1}+\sum_{i<j-1} w\left(\frac{q_i}{q_j}\right)^{1/4}(k_ik_j)^{1/2}E_{ij}^+e_{ij} \tag{2}$$

$$L^-=\sum_{i=1}^{n} k_i^{-1}e_{ii}-\sum_{i=1}^{n} w\left(\frac{q_i}{q_j}\right)^{1/4}(k_ik_{i+1})^{-1/2}X_i^-e_{i+1,i}-\sum_{i-1>j} w\left(\frac{q_i}{q_j}\right)^{1/4}(k_ik_j)^{1/2}E_{ij}^-e_{ij} \tag{3}$$

where $(e_{ij})_{kl} = \delta_{il}\delta_{jk}$ and $w = (q - q^{-1})$. It was then shown that upon suitable identification of the elements k_i with exponentials of the Cartan generators H_i, one could characterize $X_q(A_{n-1})$ as an algebra generated by the elements H_i, $X_i^{\pm}$, $(i = 1, \ldots, n)$ and satisfying the following relations:

$$[H_i, H_j] = 0 \tag{4}$$

$$[H_i, X_j^{\pm}] = \pm a_{ij} X_j^{\pm} \tag{5}$$

$$(q_i - q_{i+1}) X_i^{\pm^2} = 0 \tag{6}$$

$$[X_i^{\pm}, X_j^{\pm}] = 0 \qquad \text{if } a_{ij} = 0 \tag{7}$$

$$[X_i^{+}, X_j^{-}] = \delta_{ij} \frac{q^{H_i}\Theta_i - q^{-H_i}\Theta_i^{-1}}{q - q^{-1}} \tag{8}$$

$$q_i X_i^{\pm^2} X_{i\pm1}^{\pm} - (1 + q_i q_{i+1}) X_i^{\pm} X_{i\pm1}^{\pm} X_i^{\pm} + q_{i+1} X_{i\pm1}^{\pm} X_i^{\pm^2} = 0. \tag{9}$$

Here $\Theta_i = \exp(t_{ij}(A^{-1}))_{jk} H_k$ where $t_{ij} = \ln \omega_{ij}$ and A is the Cartan matrix of A_{n-1}, and

$$\omega_{ij} = \frac{q_i^{\delta_{ij} - \delta_{i-1,j}} q_{i+1}^{\delta_{ij} - \delta_{i+1,j}}}{q^{a_{ij}}} \tag{10}$$

is called the twisting matrix. Relations (6) are the main new features of this quantum group and equations (9) are the generalizations of the deformed Serre relations [3, 8]. This algebra was also equipped with the following Hopf structure:

$$\Delta(H_i) = H_i \otimes 1 + 1 \otimes H_i \tag{11}$$

$$\Delta(X_i^{\pm}) = q^{-H_i/2}\Theta_{ii}^{-1/2} \otimes X_i^{\pm} + X_i^{\pm} \otimes q^{H_i/2}\Theta_i^{1/2} \tag{12}$$

$$\varepsilon(H_i) = \varepsilon(X_i^{+}) = 0 \qquad \varepsilon(1) = 1 \tag{13}$$

$$S(H_i) = -H_i \qquad S(X_i^{+}) = -(q_i q_{i+1})^{1/2} X_i^{+}. \tag{14}$$

The above algebra clearly reduces to $U_q(A_{n-1})$ upon the standard choice of all $q_i = q$. However the Serre relations (9), when $q_i q_{i+1}$ become trivial identities due to (6) and hence cannot be used to reconstruct the full structure of the algebra and the q-analogues of the Poincare-Birkhoff-Wit (PBW) basis [5]. But the FRT formalism can yield the relations between the q-analogues of roots directly. For this we note that solutions of the FRT equations for (1), can neatly be summarized in the following form: for elements a, b, c, d as the entries in (row, column) (i, j), (i, l), (k, j) and (k, l) respectively, the following relations hold: elements

$$ab = q_i^{-1} ba \qquad cd = q_k^{-1} dc \tag{15}$$

$$ca = q_j ac \qquad db = q_l bd \tag{16}$$

$$bc = cb \qquad ad - da = (q^{-1} - q) bc. \tag{17}$$

This together with the ansatz (2) for L^+ leads to the following relations for the q-analogues of positive roots. (Negative roots satisfy similar relations.)

(a) The relations between Cartan generators and the positive roots:

$$[H_k, E_{i,j+1}] = (\delta_{k,i} + \delta_{k,j+1}) E_{i,j+1}. \tag{18}$$

(b) The relation between simple roots and positive non-simple roots

$$[X_i, X_{i+1}]_{q_{i+1}^{-1}} = -E_{i,i+2} \tag{19}$$

$$[X_i, E_{ij}]_{q_i} = 0 \tag{20}$$

$$[X_j, E_{ij}]_{q_j} = E_{i,j+1} \tag{21}$$

$$[X_j, E_{i,j+1}]_{q_{j+1}^{-1}} = 0 \tag{22}$$

$$[X_{i-1}, E_{ij}]_{q_i^{-1}} = 0 \tag{23}$$

where $[a, b]_q = q^{1/2}ab - q^{-1/2}ba$. These relations imply the following:

$$E_{i,j+1} = S_j \circ S_{j-1} \circ S_{j-2} \circ \ldots S_{i+1}(X_i) \tag{24}$$

where by S_i we mean $ad_{q_i}(X_i)$.

(c) The relations between positive non-simple roots

$$(q_i - q_j)E_{ij}^2 = 0 \tag{25}$$

$$[E_{ij}, E_{il}]_{q_l} = 0 \qquad j < l \tag{26}$$

$$[E_{ij}, E_{j,l}]_{q_j^{-1}} = -E_{il} \qquad i < l \tag{27}$$

$$[E_{ij}, E_{hl}] = 0 \qquad i < j < k or (i > k, j < l) \tag{28}$$

$$[E_{ij}, E_{kl}] = (q^{-1} - q)E_{il}E_{kj} \qquad j > k > i \tag{29}$$

where we have suppressed all the relations between commuting elements.

$X_q(A_{n-1})$ has two Hopf subalgebras (Borel Structures) generated respectively by H_i, X_j^+ and H_i, X_j^-. We denote them by B and B^-. The quantum double as an algebra is isomorphic to $B \otimes B'$ where B' is the dual of B with opposite comultiplication. The dual itself is denoted by B^0. For the construction of the quantum double and the R matrix, it is most convenient to redefine the generators as follows†:

$$E_{ij} \to e_{ij} = q^{H_{ij}/2} \Theta_{ij}^{1/2} E_{ij} \tag{30}$$

$$F_{ij} \to f_{ij} = q^{-H_{ij}/2} \Theta_{ij}^{-1/2} F_{ij} \tag{31}$$

where H_{ij} and Θ_{ij} stand respectively for $H_i + H_{i+1} + \ldots H_{j-1}$ and $\Pi_{h=i}^{j-1} \Theta_h$. In what follows we sometimes denote $e_{i,i+1}$ and $f_{i+1,i}$ by e_i and f_i respectively.

The q-analogue of the PBW basis for B is chosen as follows:

$$\xi^\alpha = H_1^{\alpha_1} H_2^{\alpha_2} \ldots H_{n-1}^{\alpha_{n-1}} \widehat{\prod_{i<j}} e_{ij}^{\alpha_{ij}} \tag{32}$$

where by $\widehat{\Pi}$ we mean normal ordering in the sense of [9], according to which the descendant of two roots lies between them (i.e. $[e_\alpha, e_\beta]$ lies between e_α and e_β). B^0 is generated by the elements $\eta_1, \eta_2, \ldots, \eta_{n-1}, Y_{ij}, 1 < i < n$ such that the evaluation of η_i (resp. Y_{ij}) on H_i (resp. e_{ij}) are 1 and their evaluation on any other monomial is zero. The algebra and co-algebra structure of B^0 is then defined to be [3]

$$UV(M) = U \otimes V(\Delta M) \qquad \Delta U(M \otimes N) = U(MN) \tag{33}$$

for U and $V \in B^0$, M and $N \in B$.

† We have used the notation e_{ij} for two different purposes, one as defined in (30) and the other as the basis vectors of $GL(n)$, we hope that this does not cause any confusion.

In the above evaluations one must carefully use the commutation relations to arrange every monomial occurring in the calculations in the canonical form which has been chosen for the basis of B^0. Comparison of the evaluation of UV and VU on elements of the basis, will uniquely specify the commutation relations between all pairs of elements. The coproducts are also obtained in a similar manner. From (33) it is straightforward to prove the following:

Proposition 1.

(i) $[\eta_i, \eta_j] = 0$

(ii) $[\eta_i, Y_j] = -(h\delta_{ij} + t_{jk}a_{ki})Y_j$ where $q = e^h$

(iii) $(q_i - q_{i+1})Y_i^2 = 0$

(iv) $\Delta(\eta_i) = \eta_i \otimes 1 + 1 \otimes \eta_i$

(v) $\Delta(Y_i) = 1 \otimes Y_i + Y_i \otimes e^{-a_{ij}\eta_j}$.

To complete the structure of the quantum double, we also need the commutation relation between the following pairs: (H_i, η_j), (H_i, Y_j), (e_i, η_j), and (e_i, Y_j).

The prescription for obtaining these relations can be manipulated into the following formula [10]:

$$ab = T \circ \gamma \circ Z(\Delta'^2(a) \otimes (S_0 \otimes id \otimes id)\Delta^2(b)) \tag{34}$$

where $a \in B^0$, $b \in B$, $\Delta^2 = (\Delta \otimes id)\Delta$, Δ' is the opposite co-multiplication in B^0 and S_0 is the skew antipode [3]. $Z: B^{0^{\otimes 3}} \otimes \underline{B}^{\otimes 3} \to B^0 \otimes B$ is the evaluation between the pairs $(1, 4)$ and $(3, 6)$, γ is the mutliplication map $\gamma: x \otimes y \to xy$ and T is the transposition $T(xy) = yx$.

Using (34) we can prove the following:

Proposition 2.

(i) $[\eta_i, H_j] = 0$

(ii) $[\eta_i, e_j] = (h\delta_{ij} + t_{jk}a_{ki}^{-1})e_j$

(iii) $[H_i, Y_j] = -a_{ji}Y_j$

(iv) $[Y_i, e_j] = \delta_{ij}(e^{-a_{ik}\eta_k} - q^{H_i}\Theta_i)$.

Equations (4–12) and propositions 1 and 2 completely specify the structure of the quantum double. According to Drinfeld [3], the quantum double contains the original algebra as Hopf subalgebra. This is seen in our case via the following identification:

$$\eta_i \to c_{ik}H_k \qquad Y_i \to (q - q^{-1})f_i. \tag{35}$$

Where C_{ij} is the matrix:

$$C = hA^{-1} + A^{-1}hA^{-1}. \tag{36}$$

In (36) A is the Cartan matrix of $sl(n)$, $h = \ln q$ and T is the twisting matrix (10).

The universal R matrix [3] in $B \otimes B'$ is given by

$$R = \sum \xi_\alpha \otimes \xi^\alpha \tag{37}$$

where ξ_α is the dual basis of ξ^α. The universal R matrix in $X_q(A_{n-1})$ is then obtained by applying the projection (35) to the R-matrix in the quantum double. To carry out this step we note that with the normal ordering in (33) the following basis is dual to ξ^α up to normalization.

$$\xi_\beta = \prod_{i=1}^{N-1} \eta_i^{\beta_i} \widehat{\prod_{i<j}} Y_{ij}^{\beta_{ij}} \tag{38}$$

where we mean the same ordering as in (34). To obtain the normalization factors we need the following:

Lemma 3.

(i) $\eta_i^r(H_j^s)=\delta_{ij}\delta_{rs}s!$

(ii) $Y_{ij}^n(e_{rs}^m)=\delta_{ir}\delta_{js}\delta_{mn}[n,(q_iq_j)^{-1}]!$

where $[n,\alpha]!=\Pi_{k=1}^n(\alpha^k-1)/(\alpha-1)$.

Corollary.

(a) If both q_i and q_j are untwisted (i.e. $q_i=q_j=q$) then $Y_{ij}^n(e_{ij}^n)=[n,(q)^{-2}]!$ as in the standard case [4, 10].

(b) If only one of the q_i or q_j are twisted, ($q_iq_j=-1$) then we have

$$[n,(q_iq_j)^{-1}]!=[n,-1]!=\begin{cases}0 & n>1\\ 1 & n\leqslant 1.\end{cases}$$

This is in accord with the fact that that when $q_i\neq q_j$ then $e_{ij}^2=Y_{ij}^2=0$ and monomials of e_{ij} and Y_{ij} with powers greater than 1 do not belong to the Poincare basis.

Now the normalization factors are given by

$$\xi_\alpha(\xi_\alpha)=\prod_i\alpha_i!\prod_{i<j}[\alpha_{ij},(q_iq_j)^{-1}]!\equiv N_\alpha \qquad (39)$$

from equations (32, 37–39), the final form of the universal R matrix in the quantum double is obtained:

$$R=\mathrm{e}^{\Sigma_{i=1}^{n-1}\eta_i\otimes H_i}\widehat{\prod_{\alpha_{ij}}}E_{(q_iq_j)^{-1}}(Y_{ij}\otimes e_{ij}) \qquad (40)$$

where $E_q(x)$ is the q-exponential function:

$$E_q(x)=\sum_{r=0}\frac{x^r}{[r,q]!}.$$

On passing to the quotient Hopf algebra via the projection (35), we will obtain the universal R matrix in $X_q(A_{n-1})$

$$R=\mathrm{e}^{\Sigma_{i=1}^{n-1}e_{ik}H_k\otimes H_i}\widehat{\prod_{\alpha_{ij}}}E_{(q_iq_j)^{-1}}(wf_{ij}\otimes e_{ij}). \qquad (41)$$

Note that in the standard case, we have $t_{ij}=0$ and hence $c=hA^{-1}$ and $E_{(q_iq_j)^{-1}}=E_{q^{-2}}$ which shows that the above R matrix reduces to the R matrix of $U_q(A_{n-1})$ (compare with [4, 10].

In conclusion, in the category of quasitriangular Hopf algebras, there are many more objects than the quantized enveloping algebras of Drinfeld [3] and Jimbo [8]. These are the quantum groups which are associated to the non-standard solution of Yang–Baxter equation (see [11, 12] and references therein). The quantum group $X_q(A_{n-1})$ is one of these objects. If one can clarify the relation of $X_q(A_{n-1})$ with the quantum supergroup $U_q(sl(n|m))$, then the results of the present letter can be used to construct the quantum double for the superalgebra $sl(n|m)$.

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